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Abstract

Linearized equations are set up to describe transverse waves in a relativistic plasma embedded in a magnetic field and in the presence of a non-relativistic plasma. The waves are taken to propagate at right angles to the unperturbed field. Unstable waves are found to exist when the momentum distribution is taken to be of the cosmic ray type, but anisotropic. The instability exists only when relativistic particles are present and is a consequence of the mass of a fast particle being different from its rest mass.

The maximum amplification rate is found for this case and is illustrated numerically by assuming physical values appropriate to the case of galactic cosmic ray electrons. It is shown that any anisotropy is rapidly destroyed and for cosmic ray electrons which are more than about a million years old isotropy is to be expected.

Introduction

It is well known that a relativistic charged particle moving in a circle in a uniform magnetic field emits synchrotron radiation mainly in the plane perpendicular to the magnetic field direction.

It is then of interest to consider the stability of a relativistic electron plasma, immersed in a homogeneous magnetic field, to waves which propagate normal to the unperturbed magnetic field direction.

We will consider the case where there also exists a thermal electron plasma embedded in the same magnetic field which is taken to be infinite in extent to avoid boundary effects.

We suppose that there also exists an immobile, cold, proton background plasma sufficient to preserve space charge neutrality so that the system is stable against electrostatic disturbances.

The question of electromagnetic stability of such a system is interesting since an unstable situation means that the relativistic electrons feed energy to the electromagnetic wave and consequently they become cool. On the other hand a stable situation means that the transverse wave is damped and thus the fast particles gain energy. Thus the mechanism may be either a cooling or a heating process for cosmic ray electrons and which it is will depend on the shape of the distribution function in momentum space. In this paper we propose to examine one particular type of distribution function which has been ascribed to cosmic rays. We will allow the distribution function to possess one arbitrary parameter, namely the degree of anisotropy, which can be varied in order that either stability or instability is achieved. In the case of instability a discussion of the minimum e-folding time for the wave is given and a numerical estimate is made to decide if the process is physicially significant in the case of galactic cosmic ray electrons.

2. The Dispersion Relation

We choose a Cartesian coordinate system so that the embedded magnetic field, of strength H_0, points along the direction of the x-axis and may be written $\underset{\sim}{H}_0 = H_0\,(1, 0, 0)$.

The perturbation electromagnetic potential, $\underset{\sim}{A}$, which is of infinitesimal amplitude A_0, is chosen to be

$$\underset{\sim}{A} = A_0\,(1, 0, 0)\, e^{i(kz-\omega t)}$$

Associated with this potential are perturbation electric and magnetic fields given by

$$\underset{\sim}{\mathcal{E}} = -c^{-1}\dot{\underset{\sim}{A}} = i\omega c^{-1} A_0 (1, 0, 0)\, e^{i(kz-\omega t)} \tag{1}$$

and

$$\underset{\sim}{h} = \nabla \times \underset{\sim}{A} = ikA_0 (0, 1, 0)\, e^{i(kz-\omega t)} \tag{2}$$

The relativistic Vlasov equation for a distribution function F can be written

$$\frac{\partial F}{\partial t} + \frac{c\,\underset{\sim}{\Pi}}{\sqrt{(1+\Pi^2)}} \cdot \frac{\partial F}{\partial \underset{\sim}{x}} + \frac{\epsilon}{mc}\left(\underset{\sim}{E} + \frac{\underset{\sim}{\Pi} \times \underset{\sim}{H}}{\sqrt{(1+\Pi^2)}}\right) \cdot \frac{\partial F}{\partial \underset{\sim}{\Pi}} = 0 \qquad , \qquad (3)$$

where m is the particle's rest mass, ϵ its charge, c is the velocity of light and $\underset{\sim}{\Pi}$, the normalized momentum, is given by $mc\,\underset{\sim}{\Pi} = \underset{\sim}{p}$ where $\underset{\sim}{p}$ is the actual momentum.

We linearize (3) by setting

$$F = f_0 + f$$

$$\underset{\sim}{E} = \underset{\sim}{0} + \underset{\sim}{\mathcal{E}}$$

$$\underset{\sim}{H} = \underset{\sim}{H_0} + \underset{\sim}{h}$$

where subscript '0' means equilibrium values. The linearized Vlasov equation can then be written

$$\frac{\partial f}{\partial t} + \frac{c\,\underset{\sim}{\Pi}}{\sqrt{(1+\Pi^2)}} \cdot \frac{\partial f}{\partial \underset{\sim}{x}} + \frac{\epsilon\,(\underset{\sim}{\Pi} \times \underset{\sim}{H_0})}{mc\sqrt{(1+\Pi^2)}} \cdot \frac{\partial f}{\partial \underset{\sim}{\Pi}} = -\frac{\epsilon}{mc}\left(\underset{\sim}{\mathcal{E}} + \frac{\underset{\sim}{\Pi} \times \underset{\sim}{h}}{\sqrt{(1+\Pi^2)}}\right) \cdot \frac{\partial f_0}{\partial \underset{\sim}{\Pi}} \qquad (4)$$

As is usual we have neglected any electrostatic electric field.

Changing to cylindrical momentum coordinates defined by

$$\Pi_x = \Pi_{\parallel}\,, \quad \Pi_y = \Pi_{\perp} \sin\theta\,, \quad \Pi_z = \Pi_{\perp} \cos\theta$$

so that

$$\frac{\partial}{\partial \Pi_y} = \sin\theta \frac{\partial}{\partial \Pi_{\perp}} + \frac{\cos\theta}{\Pi_{\perp}} \frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial \Pi_z} = \cos\theta \frac{\partial}{\partial \Pi_{\perp}} - \frac{\sin\theta}{\Pi_{\perp}} \frac{\partial}{\partial \theta}$$

and making use of (1) and (2), we see that (4) can be written

$$\left(\frac{ck\Pi_\perp\cos\theta}{\sqrt{(1+\Pi^2)}}-\omega\right)\varphi-\frac{i\omega_L}{\sqrt{(1+\Pi^2)}}\frac{\partial\varphi}{\partial\theta}=-\frac{\epsilon A_0}{mc}\left[\frac{\omega}{c}\frac{\partial f_0}{\partial\Pi_\parallel}+\frac{k\cos\theta}{\sqrt{(1+\Pi^2)}}\left(\Pi_\parallel\frac{\partial f_0}{\partial\Pi_\perp}-\Pi_\perp\frac{\partial f_0}{\partial\Pi_\parallel}\right)\right], \tag{5}$$

where $f=\varphi\, e^{i(kz-\omega t)}$ and $\omega_L mc=\epsilon H_0$. We have also assumed $\frac{\partial f_0}{\partial\theta}=0$ as is reasonable in any physical situation.

It is easy to show that the solution to (5) is given by

$$\varphi=\frac{-\epsilon A_0}{mc^2}e^{-\frac{ick\Pi_\perp\sin\theta}{\omega_L}}\sum_{s=-\infty}^{\infty}\frac{J_s\left(\frac{ck\Pi_\perp}{\omega_L}\right)e^{is\theta}}{\left(s\omega_L-\omega\sqrt{(1+\Pi^2)}\right)}\left[\omega\sqrt{(1+\Pi^2)}\frac{\partial f_0}{\partial\Pi_\parallel}+\frac{s\omega_L}{\Pi_\perp}\left(\Pi_\parallel\frac{\partial f_0}{\partial\Pi_\perp}-\Pi_\perp\frac{\partial f_0}{\partial\Pi_\parallel}\right)\right]. \tag{6}$$

We must also satisfy the Maxwell equation

$$\nabla^2\underset{\sim}{A}-c^{-2}\frac{\partial^2\underset{\sim}{A}}{\partial t^2}=-4\pi\epsilon\int\frac{\underset{\sim}{\Pi}\, f\, d^3\Pi}{\sqrt{(1+\Pi^2)}},$$

which can be written

$$\left(k^2-\omega^2/c^2\right)A_0=4\pi\epsilon\int\frac{\Pi_x\,\varphi\, d^3\Pi}{\sqrt{(1+\Pi^2)}}. \tag{7}$$

Substituting for φ from (6) and performing the integral over θ, enables (7) to be written

$$k^2-\omega^2/c^2=-\frac{8\pi^2\epsilon^2}{mc^2}\sum_{s=-\infty}^{\infty}\int_0^\infty d\Pi_\perp\int_{-\infty}^{\infty}\frac{\Pi_\parallel\Pi_\perp J_s^2\left(\frac{ck\Pi_\perp}{\omega_L}\right)}{\sqrt{(1+\Pi^2)}\left(s\omega_L-\omega\sqrt{(1+\Pi^2)}\right)}\left[\omega\sqrt{(1+\Pi^2)}\frac{\partial f_0}{\partial\Pi_\parallel}+\frac{s\omega_L}{\Pi_\perp}\left(\Pi_\parallel\frac{\partial f_0}{\partial\Pi_\perp}-\Pi_\perp\frac{\partial f_0}{\partial\Pi_\parallel}\right)\right]d\Pi_\parallel. \tag{8}$$

Transforming to spherical momentum coordinates defined by $\Pi_{\parallel} = \Pi\cos\psi$, $\Pi_{\perp} = \Pi\sin\psi$ we see that (8) can be written

$$k^2 - \omega^2/c^2 = -\frac{8\pi^2\epsilon^2}{mc^2}\sum_{s=-\infty}^{\infty}\int_0^{\pi} d\psi \int_0^{\infty} \frac{\Pi^3 \sin\psi\cos\psi\, J_s^2\left(\frac{ck\Pi\sin\psi}{\omega_L}\right)}{\left(s\omega_L - \omega\sqrt{(1+\Pi^2)}\right)\sqrt{(1+\Pi^2)}} \times$$

$$\left[\omega\sqrt{(1+\Pi^2)}\left(\cos\psi\frac{\partial f_0}{\partial \Pi} - \frac{\sin\psi}{\Pi}\frac{\partial f_0}{\partial\psi}\right) + \frac{s\omega_L}{\Pi\sin\psi}\frac{\partial f_0}{\partial\psi}\right] d\Pi \; . \tag{9}$$

This can be re-arranged to give

$$k^2 - \omega^2/c^2 + \frac{8\pi^2\epsilon^2}{mc^2}\int_0^{\pi} d\psi \int_0^{\infty} \frac{\Pi^2\cos\psi}{\sqrt{(1+\Pi^2)}}\frac{\partial f_0}{\partial\psi}\sum_{s=-\infty}^{\infty} J_s^2\left(c\frac{k\Pi\sin\psi}{\omega_L}\right) d\Pi$$

$$= -\frac{8\pi^2\epsilon^2\omega}{mc^2}\sum_{s=-\infty}^{\infty}\int_0^{\pi} d\psi \int_0^{\infty} \frac{\Pi^3\sin\psi\cos^2\psi\, J_s^2\left(\frac{ck\Pi\sin\psi}{\omega_L}\right)}{\left(s\omega_L - \omega\sqrt{(1+\Pi^2)}\right)}\left(\frac{\partial f_0}{\partial\Pi} + \frac{\cot\psi}{\Pi}\frac{\partial f_0}{\partial\psi}\right) . \tag{10}$$

Now it is well known that

$$\sum_{s=-\infty}^{\infty} J_s^2(x) = 1$$

for all values of x. Thus the third term on the left hand side of (10) can be written

$$\frac{8\pi^2\epsilon^2}{mc^2}\int_0^{\pi}\cos\psi\, d\psi \int_0^{\infty} \Pi^2(1+\Pi^2)^{-1/2}\frac{\partial f_0}{\partial\psi}\, d\Pi$$

On the right hand side of (10) we see that we have a factor inside the double integral of the form

$$T = \sum_{s=-\infty}^{\infty} \frac{J_s^2(a)}{(s-b)} \tag{11}$$

Upon multiplying each term of (11) by $(s+b)/(s+b)$ and noting that $J_{-s}^2(x) = J_s^2(x)$ we see that (11) becomes

$$T = b \sum_{s=-\infty}^{\infty} \frac{J_s^2(a)}{(s^2-b^2)} .$$

Using the results given in the appendix it can be shown that

$$T = -\pi \operatorname{cosec}(\pi b) J_b(a) J_{-b}(a) .$$

Thus we can simplify (10) to obtain

$$k^2 - \omega^2/c^2 + \frac{8\pi^2\epsilon^2}{mc^2} \int_0^{\pi} \cos\psi \, d\psi \int_0^{\infty} \frac{\Pi^2}{\sqrt{(1+\Pi^2)}} \frac{\partial f_0}{\partial \psi} \, d\Pi$$

$$= \frac{8\pi^2\epsilon^2\omega}{mc^2\omega_L} \int_0^{\pi} d\psi \int_0^{\infty} \Pi^3 \sin\psi \cos^2\psi \operatorname{cosec}\left(\frac{\pi\omega}{\omega_L}\sqrt{(1+\Pi^2)}\right) J_{\frac{\omega}{\omega_L}\sqrt{(1+\Pi^2)}}\left(\frac{ck\Pi\sin\psi}{\omega_L}\right) J_{-\frac{\omega}{\omega_L}\sqrt{(1+\Pi^2)}}\left(\frac{ck\Pi\sin\psi}{\omega_L}\right) \times$$

$$\left(\frac{\partial f_0}{\partial \Pi} + \frac{\cot\psi}{\Pi}\frac{\partial f_0}{\partial \psi}\right) d\Pi . \tag{12}$$

We will refer to either (10) or (12) as the dispersion relation since they define the spread of k with ω. Since we are looking for the temporal stability of the system we define k to be real and positive. The necessary and sufficient condition for an unstable situation to exist is that the imaginary part of (12) vanish for some real positive ω (Penrose, 1960).

To find the imaginary part of (12) it is necessary to consider the position of the poles which occur in the Π integral. It is easily seen that poles occur in the Π integral whenever

$$\Pi = \omega^{-1}\sqrt{(s^2\omega_L^2 - \omega^2)}\ .$$

For ω real and positive it is obvious that the poles will lie on the real Π axis for all $s \geqslant \omega_L/\omega$. . For $s < \omega/\omega_L$ the poles lie on the imaginary Π axis.

Writing

$$X = \Pi - \omega^{-1}\sqrt{(s^2\omega_L^2 - \omega^2)}$$

we see that as $\mathrm{Im}(\omega) \to 0$ from above, we must understand

$$\frac{1}{X} = P\left(\frac{1}{X}\right) - i\pi\delta(X)$$

for all $s \geqslant \omega/\omega_L$, where we have initially defined ω to lie in the upper half complex plane. For a thorough discussion of this point we refer the reader to Jackson (1958).

Thus for ω real and positive we see that (12) becomes

$$k^2-\omega^2/c^2+\frac{8\pi^2\epsilon^2}{mc^2}\int_0^\pi \cos\psi\, d\psi\int_0^\infty \Pi^2(1+\Pi^2)^{-1/2}\frac{\partial f_0}{\partial\psi}\,d\Pi$$

$$-\frac{8\pi^3\epsilon^2\omega}{mc^2\omega_L}\int_0^\pi d\psi\, P\!\!\int_0^\infty \sin\psi\cos^2\psi\,\Pi^3\operatorname{cosec}\left(\pi\frac{\omega}{\omega_L}\sqrt{(1+\Pi^2)}\right)J_{\frac{\omega}{\omega_L}\sqrt{(1+\Pi^2)}}\left(\frac{ck\Pi\sin\psi}{\omega_L}\right)J_{-\frac{\omega}{\omega_L}\sqrt{(1+\Pi^2)}}\left(\frac{ck\Pi\sin\psi}{\omega_L}\right)\left(\frac{\partial f_0}{\partial\Pi}+\frac{\cot\psi}{\Pi}\frac{\partial f_0}{\partial\psi}\right)d\Pi$$

$$=\frac{-4i\pi^3\epsilon^2\omega_L}{mc^2\omega^3}\sum_{s\geq\omega/\omega_L}^{\infty}s(s^2\omega_L^2-\omega^2)\int_0^\pi \sin\psi\cos^2\psi\, J_s^2\left(\frac{ck\sin\psi\sqrt{(s^2\omega_L^2-\omega^2)}}{\omega\omega_L}\right)\left(\frac{\partial f_0}{\partial\Pi}+\frac{\omega\cot\psi}{\sqrt{(s^2\omega_L^2-\omega^2)}}\frac{\partial f_0}{\partial\psi}\right)\Bigg|_{\Pi=\omega^{-1}\sqrt{(s^2\omega_L^2-\omega^2)}} d\psi\ . \tag{13}$$

If the equilibrium distribution function is isotropic in momentum space and monotonically decreasing with increasing Π we see that the imaginary part of (13) is always negative and consequently this situation is always stable.

We can also note that if only thermal particles are present so that we can replace the relativistic factor $\sqrt{(1+\Pi^2)}$ by unity then the imaginary part of (13) is identically zero for all real ω values and thus this situation would also be stable. In such a case for real ω , k^2 would change sign as ω passed through $s\omega_L$ and would become, say, positive infinite and negative infinite for $\omega = s\omega_L \pm 0$ respectively.

Thus waves would either propagate or be heavily damped depending on which side of $s\omega_L$ the frequency, ω , lay. Hence only for relativistic particles is it possible to have an unstable situation.

Let us therefore consider the case where an anisotropic relativistic electron plasma is present and also a cold thermal electron plasma. We shall also assume that a cold background proton plasma is present which serves to satisfy space charge neutrality everywhere.

3. The Equilibrium Distribution Function

While there are many possible relativistic electron distributions which one can use in (12) we wish to choose one which has some probability of being realized in nature. From cosmic ray measurements an inverse power law spectrum seems to fit the data reasonably well, so for the relativistic electrons we choose

$$f_0 = \varsigma \Pi^{-(\gamma+2)} (1+\alpha \cos^2\psi)\, S(\Pi - \Pi_0) \qquad . \tag{14}$$

Here γ is a fixed index and α we call the degree of anisotropy. The step function $S(x)$ is defined by

$$\begin{aligned} S(x) &= 1 \qquad x > 0 \\ &= 0 \qquad x < 0 \end{aligned}$$

It is clear that with an inverse power law such a lower momentum cut-off is essential in order to keep the number density, ρ, of relativistic particles finite. Provided γ is large enough (> 2) no upper momentum cut-off is required to keep the energy density of cosmic ray electrons finite.

The constant ς is chosen so that

$$2\pi \int_0^\infty d\Pi \int_0^\pi f_0 \Pi^2 \sin\psi \, d\psi = \rho \quad . \tag{15}$$

Let us further assume that, while the relativistic electrons provide the imaginary part of (13), the thermal particles have a much higher number density than the relativistic particles and hence provide the real part of (13). Thus it is the relativistic particles which provide the energy exchange with the wave but it is the thermal particles which define the wavenumber at which this exchange occurs.

We treat the thermal particles as being completely cold so that their distribution function can be written

$$f_0 = \frac{N\,\delta(\Pi)\,\delta(\psi)}{2\pi\,\Pi^2 \sin\psi} \tag{16}$$

where N is the number density of cold electrons. Substituting (16) into (13) and evaluating the integrals we see that the real part can be written

$$k^2 = (\omega^2 - \omega_p^2)\,c^{-2} \quad , \tag{17}$$

where the plasma frequency for cold particles is defined by

$$\omega_p^2 = 4\pi N e^2/m \quad .$$

In terms of the phase refractive index of the wave, n , we have

$$c^2k^2/\omega^2 = n^2 = 1 - (\omega_p/\omega)^2 \tag{18}$$

Thus the usual condition for a propagating wave to exist, namely $\omega > \omega_p$, is recovered in the limit of the cold electron approximation.

Let us now turn our attention to the imaginary part of (13). Writing

$$k^2 = M(\omega) + i\,N(\omega)$$

where M and N are real functions of the real variable ω we see that, since k^2 is real, we have

$$N(\omega) = \frac{4\pi^3\epsilon^3\omega_L}{mc^2\omega^3}\sum_{s\geq\eta\omega_L}^{\infty} s(s^2\omega_L^2-\omega^2)\int_0^{\pi}\sin\psi\cos^2\psi\, J_s^2\left(\frac{ck\Pi\sin\psi}{\omega_L}\right)\left(\frac{\partial f_0}{\partial\Pi}+\frac{\cot\psi}{\Pi}\frac{\partial f_0}{\partial\psi}\right)\Bigg|_{\Pi=\omega^{-1}\sqrt{(s^2\omega_L^2-\omega^2)}} d\psi \,, \tag{19}$$

and with f_0 given by (14) we must demand that $N(\omega) = 0$ for some real positive ω if an unstable situation is to develop.

Substituting for (14) in (19) we see that

$$N(\omega) = \frac{4\pi^3\epsilon^3\omega_L\,\zeta\,\omega^{\gamma-1}}{mc^2}\sum_{s\geq\frac{\omega}{\omega_L}\sqrt{(1+\Pi_0^2)}}^{\infty}\frac{s}{(s^2\omega_L^2-\omega^2)^{\gamma/2}}\int_0^{\pi}\sin\psi\cos^2\psi\; J_s^2\left(\frac{n\sin\psi\sqrt{(s^2\omega_L^2-\omega^2)}}{\omega_L}\right) \times$$

$$\left\{\left[-\omega(\gamma+2)(s^2\omega_L^2-\omega^2)^{-1/2}+(1+\Pi_0^2)^{1/2}\Pi_0^{-1}\delta\left(s-\frac{\omega}{\omega_L}\sqrt{(1+\Pi_0^2)}\right)\right]+\alpha\cos^2\psi\left[\frac{-\omega(\gamma+4)}{\sqrt{(s^2\omega_L^2-\omega^2)}}+\frac{\sqrt{(1+\Pi_0^2)}}{\Pi_0}\delta\left(s-\frac{\omega}{\omega_L}\sqrt{(1+\Pi_0^2)}\right)\right]\right\}. \tag{20}$$

Assuming that Π_0 is sufficiently large compared to unity so that $\sqrt{(1+\Pi_0^2)} \simeq \Pi_0$ and $s\omega_L \gg \omega$ we see that (20) becomes

$$N(\omega) \simeq \frac{4\pi^3\epsilon^2\zeta(\omega/\omega_L)^{\gamma-1}}{mc^2}\sum_{s\geq\frac{\omega\Pi_0}{\omega_L}}^{\infty} s^{1-\gamma}\int_0^{\pi}\sin\psi\cos^2\psi\, J_s^2(ns\sin\psi) \times$$

$$\left\{\left[-\frac{\omega(\gamma+2)}{s\omega_L}+\delta\left(s-\frac{\omega\Pi_0}{\omega_L}\right)\right]+\alpha\cos^2\psi\left[\frac{-\omega(\gamma+4)}{s\omega_L}+\delta\left(s-\frac{\omega\Pi_0}{\omega_L}\right)\right]\right\} \,. \tag{21}$$

Since we know that $\omega \gg \omega_p$ and Π_0 is reasonably large compared to unity we can replace the sum over S by an integral without appreciable error. Writing $s = (\omega/\omega_L)x$ and letting $(\omega/\omega_L) = \Omega$ we see that (21) becomes

$$N(\omega) \simeq \frac{4\pi^3\epsilon^2 S\Omega}{mc^2}\int_{\Pi_0}^{\infty} x^{1-\gamma}dx \int_0^{\pi} \sin\psi\cos^2\psi \, J_{\Omega x}^2(n\sin\psi\,\Omega x) \times$$

$$\left\{ \left[-\frac{(\gamma+2)}{x} + \Omega^{-1}\delta(x-\Pi_0)\right] + \alpha\cos^2\psi\left[-\frac{(\gamma+4)}{x} + \frac{\delta(x-\Pi_0)}{\Omega}\right]\right\} d\psi \quad . \tag{22}$$

Now $n < 1$ and thus $n\sin\psi < 1$. Also since $\omega \gg \omega_p$ and $\Pi_0 \gtrsim 1$ we can use the Carlini approximation to the Bessel function so that

$$J_\nu(\nu\beta) \simeq \left(2\pi\nu\sqrt{(1-\beta^2)}\right)^{-1/2}\left(\frac{1-\sqrt{(1-\beta^2)}}{1+\sqrt{(1-\beta^2)}}\right)^{\nu/2} e^{\nu\sqrt{(1-\beta^2)}} \, , \tag{23}$$

where $\nu = \Omega x$ and $\beta = n\sin\psi$.

Upon using this approximation in (22) we see that

$$N(\omega) \simeq \frac{2\pi^2\epsilon^2 S}{mc^2}\int_{\Pi_0}^{\infty} x^{-\gamma}dx \int_0^{\pi}\frac{\sin\psi\cos^2\psi}{\sqrt{(1-n^2\sin^2\psi)}}\left(\frac{1-\sqrt{(1-n^2\sin^2\psi)}}{1+\sqrt{(1-n^2\sin^2\psi)}}\right)^{\Omega x} e^{2\Omega x\sqrt{(1-n^2\sin^2\psi)}} \times$$

$$\left\{\left[-(\gamma+2)x^{-1} + \Omega^{-1}\delta(x-\Pi_0)\right] + \alpha\cos^2\psi\left[-(\gamma+4)x^{-1} + \Omega^{-1}\delta(x-\Pi_0)\right]\right\} d\psi \, . \tag{24}$$

Changing the order of integration in (24) we see that

$$N(\omega) \simeq \frac{2\pi^2\epsilon^2 S}{mc^2}\int_0^{\pi}\frac{\sin\psi\cos^2\psi\,d\psi}{\sqrt{(1-n^2\sin^2\psi)}}\int_{\Pi_0}^{\infty}\frac{e^{-ax}}{x^{\gamma}}\left[\Omega^{-1}\delta(x-\Pi_0)(1+\alpha\cos^2\psi) - x^{-1}(\gamma+2+\alpha\cos^2\psi(\gamma+4))\right]dx. \tag{25}$$

where

$$a = \Omega \ln\left[\left(\frac{1+\sqrt{(1-n^2\sin^2\psi)}}{1-\sqrt{(1-n^2\sin^2\psi)}}\right)e^{-2\sqrt{(1-n^2\sin^2\psi)}}\right] \tag{26}$$

If the x-integral is to converge it is clear that we require $a > 0$. Since $n < 1$ the smallest value of a occurs when $\sin\psi = 1$ and at this value we require

$$\ln\left(1+\sqrt{(1-n^2)}\right) - \ln\left(1-\sqrt{(1-n^2)}\right) > 2\sqrt{(1-n^2)} \tag{27}$$

Remembering that $n^2 = 1-(\omega_p/\omega)^2$ and $\omega > \omega_p$ we see that (27) is always obeyed. Thus the integral over x converges.

While the x integral cannot be performed exactly an estimate of the value can be made in the following manner. Consider

$$I(r) = \int_{\Pi_0}^{\infty}\frac{e^{-ax}}{x^r}dx$$

where $r > 1$, $\Pi_0 \gg 1$. It is clear that for $x \gtrsim \Pi_0 + a^{-1}$ the integrand rapidly becomes very small compared with its value at Π_0. So that to a reasonable degree of accuracy we can write

$$I(r) \propto \int_{\Pi_0}^{\Pi_0 + a^{-1}} \frac{e^{-ax}}{x^r}\,dx \simeq \frac{e^{-a\Pi_0}\left(1-e^{-1}\right)}{a\,\Pi_0^{\,r}} \qquad , \tag{28}$$

Making use of (28) in (25) we see that

$$N(\omega) \simeq \frac{-4\pi^2\epsilon^2\zeta}{mc^2}\int_0^{\pi/2} \frac{\sin\psi\cos^2\psi}{\sqrt{(1-n^2\sin^2\psi)}}\left(\frac{1-\sqrt{(1-n^2\sin^2\psi)}}{1+\sqrt{(1-n^2\sin^2\psi)}}\right)^{\Omega\Pi_0} e^{2\Omega\Pi_0\sqrt{(1-n^2\sin^2\psi)}} \left[\frac{\left(\gamma+2+\alpha\cos^2\psi(\gamma+4)\right)(1-e^{-1})}{\Pi_0^{\gamma+1}a} - \frac{\left(1+\alpha\cos^2\psi\right)}{\Omega\,\Pi_0^{\,\gamma}}\right], \tag{29}$$

where a is given by (26).

Changing variables in (29) through the relation

$$\cos\psi = \frac{\sqrt{(1-n^2)}}{n}\left(\frac{\xi^2}{1-n^2}-1\right)^{1/2}$$

we see that (29) can be written

$$N(\omega) \simeq \frac{-4\pi^2\epsilon^2\zeta}{n^5 mc^2\Pi_0^{\gamma}\Omega}\int_{\sqrt{(1-n^2)}}^{1}\left(\frac{1-\xi}{1+\xi}\right)^{\Omega\Pi_0} e^{2\Omega\Pi_0\xi}\left[\frac{(1-e^{-1})\left((\gamma+2)+\alpha n^{-2}(\gamma+4)(\xi^2+n^2-1)\right)}{\Pi_0\ln\left(\frac{1+\xi}{1-\xi}e^{-2\xi}\right)} - \left(1+\alpha n^{-2}(\xi^2+n^2-1)\right)\right](\xi^2+n^2-1)^{1/2}\,d\xi \,. \tag{30}$$

It is obvious by inspection that (30) possesses a sharp maximum in the range $\sqrt{(1-n^2)} < \xi < 1$. Thus we can gainfully employ the method of steepest descents to evaluate the integral.

The maximum occurs when

$$\xi \simeq \sqrt{(1-n^2)}\left[1 + \frac{r n^2}{4\Omega\Pi_0(1-n^2)^{3/2}}\right] \qquad , \tag{31}$$

where r is the power of $(\xi^2+n^2-1)^{1/2}$ which occurs in the integrand of (30) and this result is approximately true provided

$$r n^2 \ll 4\Omega\Pi_0 (1-n^2)^{3/2} . \tag{32}$$

Then the method of steepest descents yields

$$N(\omega) \simeq \frac{-4\pi^{5/2}\epsilon^2 \zeta (1-n^2)^{1/4} e^{\frac{n^2}{2(1-n^2)}}}{\Omega n^2 m c^2 \Pi_0^{\gamma} (2\Omega\Pi_0)^{3/2}} \left(\frac{1-\sqrt{(1-n^2)}}{1+\sqrt{(1-n^2)}}\right)^{\Omega\Pi_0} e^{2\Omega\Pi_0\sqrt{(1-n^2)}} \left[\left(\frac{(\gamma+2)(1-e^{-1})}{\Pi_0 \ln A} - 1\right) + \frac{9\alpha e^{\frac{n^2}{(1-n^2)}}}{2\Omega\Pi_0\sqrt{(1-n^2)}}\left(\frac{(\gamma+4)(1-e^{-1})}{\Pi_0 \ln A} - 1\right)\right], \tag{33}$$

where $A = \frac{1+\sqrt{(1-n^2)}}{1-\sqrt{(1-n^2)}} \, e^{2\sqrt{(1-n^2)}}$.

It is clear that if $n \simeq 0$ then $N(\omega)$ is extremely small. Thus the situation which is of physical interest is the case where $n \simeq 1$. In such a situation $|\ln A| \ll 1$ so that we can write approximately

$$N(\omega) \simeq \frac{-\sqrt{2}\,\pi^{5/2}\epsilon^2 \zeta (1-e^{-1})\, e^{\frac{n^2}{2(1-n^2)}} (1-n^2)^{1/4}}{\Omega^2 \Pi_0^{\gamma+2} n^2 m c^2 A^{\Pi_0\Omega} \ln A}\left[(\gamma+2) + \frac{9\alpha(\gamma+4)\, e^{\frac{n^2}{(1-n^2)}}}{2\Omega\Pi_0\sqrt{(1-n^2)}}\right] . \tag{34}$$

It is clear from (34) that the imaginary part of the dispersion relation vanishes only if $\alpha < 0$.

Setting $\alpha = -\beta\ (\beta > 0)$ we see that the Penrose criterion is satisfied when

$$\omega^2 \simeq \omega_p^2 \ln\left(\frac{2\Pi_0\omega_p e(\gamma+2)}{9(\gamma+4)\omega_L\beta}\right) \equiv \omega_0^2, \text{ say.} \tag{35}$$

If this approximation is to be valid we require $k^2 > 0$ and $\omega > \omega_p$.

From (35) we see that this requires

$$\beta << \frac{2\Pi_0 \omega_p (\gamma+2)}{9(\gamma+4)\omega_L} \quad . \tag{36}$$

Also from (32) we must have

$$\beta >> \frac{2\Pi_0 \omega_p (\gamma+2)}{9(\gamma+4)\omega_L} \exp -\left(\frac{4\Pi_0 \omega_p}{3\omega_L}\right) \quad . \tag{37}$$

We will see later in the section on numerical estimates that (36) and (37) are extremely lax conditions on the permitted values of β.

With these restrictions we can approximate $N(\omega)$ further to obtain

$$N(\omega) \simeq -\frac{27\sqrt{2}\,(\gamma+4)\pi^{5/2} e^{-3/2}(1-e^{-1})\zeta\,\omega_L^{7/2}\epsilon^2 \beta\, e^{\varphi^2/2}\left(e^{\varphi_0^2}-e^{\varphi^2}\right)}{4\Pi_0^{\gamma+7/2}\,\omega_p^{7/2}\, mc^2} \tag{38}$$

where $\varphi = \omega/\omega_p$ and $\varphi_0 = \omega_0/\omega_p$.

Since the wave interacts with the particles at multiples of the cyclotron frequency we see that the interaction is mainly with the perpendicular momentum of the particles. Thus in the case where $\alpha > 0$ we have Landau damping of all waves and as a consequence the energy associated with the wave <u>increases</u> the perpendicular pressure. Thus the plasma tends to isotropy.

In the case where $\alpha < 0$ the perpendicular pressure is <u>decreased</u>

since the waves abstract more energy from the perpendicular component of motion than the parallel component. Since we initially had a greater perpendicular pressure than parallel pressure we see that the plasma again tends to isotropy.

4. Amplification Rates

Under the approximation we have made it has been shown that an unstable situation can exist for $\alpha < 0$. We must now consider the e-folding time to see if the process can have physical significance.

In general a disturbance of the form $e^{i(kz-\omega t)}$ when applied to the linearized equations yields a dispersion relation which can be put in the form

$$k^2 = J(\omega) + i\,K(\omega) \tag{39}$$

where J and K are real functions of the complex variable ω , and k^2 is positive definite.

Writing $\omega = \omega_r + i\,\delta$ we see that

$$k^2 = J(\omega_r, \delta) \quad , \tag{40a}$$

$$0 = K(\omega_r, \delta) \quad . \tag{40b}$$

Assuming a priori that $\delta \ll \omega_p$ we can expand (40) in a Taylor series. Thus

$$k^2 \simeq J(\omega, 0)$$

$$0 \simeq K(\omega,0) + \left.\frac{\partial K(\omega,\delta)}{\partial(\delta/\omega_p)}\right|_{\delta=0} (\delta/\omega_p) + \cdots \tag{41}$$

Since k^2 is meromorphic in ω and δ we see from Cauchy's relations that

$$\left.\frac{\partial K(\omega,\delta)}{\partial(\delta/\omega_p)}\right|_{\delta=0} = \left.\frac{\partial J(\omega,\delta)}{\partial(\omega/\omega_p)}\right|_{\delta=0} = \omega_p \frac{\partial J(\omega,0)}{\partial \omega} \quad . \tag{42}$$

Keeping only terms up to first order in (δ/ω_p) in (41) and using (42) we see that

$$\delta \simeq -K(\omega,0) \Big/ \frac{\partial J(\omega,0)}{\partial \omega} \quad . \tag{43}$$

Now we have already evaluated $K(\omega,0)\,(\equiv N(\omega))$ and $J(\omega,0)\,(\equiv(\omega^2-\omega_p^2)c^{-2})$ in order to satisfy the Penrose condition. Thus it is a simple matter to write down δ and upon so doing we find that

$$\delta \simeq \frac{27\sqrt{2}\,(\gamma+4)\pi^{5/2} e^{-3/2}(1-e^{-1})\,\omega_L^{7/2} \epsilon^2 \zeta \beta \varphi^{-1} e^{\varphi/2}(e^{\varphi_0^2}-e^{\varphi^2})}{8\,\Pi_0^{\gamma+7/2}\,\omega_p^{9/2}\, m} \quad . \tag{44}$$

From (44) it can easily be seen that δ possesses a maximum close to $\varphi = \varphi_0$ and at the maximum

$$\delta_{max} \simeq \frac{3\sqrt{2}\,\pi^{5/2} e^{-1/2}(1-e^{-1})\,\omega_L^{5/2}\epsilon^{2}\zeta(\gamma+2)}{16\,\Pi_0^{\gamma+5/2}\omega_p^{7/2} m\left[\ln\left(\frac{2\Pi_0\omega_p e(\gamma+2)}{9(\gamma+4)\omega_L\beta}\right)\right]^{1/2}} \quad . \qquad (45)$$

5. Numerical Estimates

To obtain some idea of the numerical size of the instability in order to see if it can be physically significant we apply the above calculation to the case of galactic cosmic ray electrons.

The parameters required are

(i) H_0 - this has been estimated to be about $5 \cdot 10^{-6}\Gamma$ (Gardner and Davies, 1965).

(ii) N - the neutral hydrogen number density has been estimated to be ~ 1 cm^{-3} of which perhaps 2% is ionized (Westerhout, 1957; Wilson, 1963) thus giving $N \sim 2 \cdot 10^{-2}$ cm^{-3}.

(iii) γ - from observations of cosmic ray protons this has a value of about 2.5 . If we assume that the cosmic ray electrons possess a similar spectrum, they also have $\gamma \simeq 2.5$.

(iv) Π_0 - for cosmic ray protons this has the value $\Pi_0 \sim 1.5$. Since we require $\Pi_0^2 \gg 1$ for the calculation to be valid we will assume $\Pi_0 \simeq 5$ for the fast electrons.

(v) ρ —. the energy density of cosmic ray protons gives

$$\mathcal{E} \simeq 2 m_p c^2(\gamma_p - 1)\rho_p \Pi_{0p} \; ,$$

where $\mathcal{E} \simeq 10^{-12}$ ergs cm^{-3} and the suffix 'p' stands for protons. If we assume that there is equipartition of energy between the cosmic ray protons and electrons then

$$\rho \simeq \frac{\mathcal{E}}{4mc^2(\gamma-1)\Pi_0} \simeq 6{\cdot}7.\ 10^{-8}\,cm^{-3},$$

Employing the above parameters in (45) we see that

$$\delta_{max}^{-1} = \tau_{min} \simeq 1{\cdot}3.10^3 \left[\ln\left(1{\cdot}3.10^2 \beta^{-1}\right)\right]^{1/2} \quad yrs. \tag{46}$$

It is clear from (46) that as $\beta \to 0$ so $\tau_{min} \to \infty$; so that the closer the plasma is to isotropy the longer it takes for the wave to abstract energy from the plasma.

It is well known that the observed cosmic ray protons are isotropic to better than 1% (Greisen, 1956) so that a β of 10^{-4} is certainly well below this observational upper limit. This value of β leads to an e-folding time of

$$\tau_{min} \simeq 4.10^3\,yrs. \tag{47}$$

In increasing its energy by a factor of 5 or so in this time the wave has drawn mainly on the perpendicular component of motion of the relativistic plasma. Then the fast electrons become more nearly isotropic. For example if we assume that cosmic ray electrons are at least 10^8 years old on the average, then we expect that isotropization will be very nearly complete with this β value.

One of the main drawbacks of a linearized calculation is that no account of the energy loss of the fast particles is given. Thus we do not know whether the relativistic electrons lose so much energy during isotropization that they cease to be relativistic, or whether the total particle energy loss is so small it can be neglected.

We wish to point out that the important consequence of instability is that the plasma rapidly becomes isotropic if the anisotropy is reasonably large.

With the parameter given we see that the conditions which we must satisfy, namely

$$a) \quad \beta << \frac{2\Pi_0 \omega_p (\gamma+2)}{9(\gamma+4)\omega_L} ,$$

$$b) \quad \beta >> \frac{2\Pi_0 \omega_p (\gamma+2)}{9(\gamma+4)\omega_L} \exp - \left(\frac{4\Pi_0 \omega_p}{3\omega_L} \right) ,$$

become

$$a) \quad \beta << 76 ,$$

$$b) \quad \beta >> 76\, e^{-150} .$$

With $\beta \simeq 10^{-4}$ it is clear that both of these conditions are met and consequently the approximate formulae for $N(\omega)$ and δ are valid.

Associated with the most unstable wave are a frequency, f_{max}, and a wavelength, λ_{max}, which, for $\beta = 10^{-4}$, are given by

$$f_{max} \simeq 5 \ kc/s .$$

and

$$\lambda_{max} \simeq 6.10^6 cm.$$

Thus the instability occurs on a very small scale compared with the size of the galaxy.

6. Conclusion

Using a simple relativistic electron momentum distribution function, in conjunction with a cold electron background, we have shown that the system is unstable against waves propagating perpendicular to the embedded magnetic field provided only that the perpendicular cosmic ray electron pressure is in excess of the parallel cosmic ray electron pressure.

Assuming numerical parameters which are representative of the values expected in the galaxy, we find that the instability is sufficiently rapid to produce isotropy of the distribution function in a very short time compared to the age of the cosmic ray electrons. For example, with an age of 10^8 years and an anisotropy of 10^{-4} we find an e-folding time which is only $\sim 10^{-5}$ of the age. For higher degrees of anisotropy the isotropization of the relativistic electrons proceeds even more rapidly.

We conclude that this process is an extremely powerful one for producing isotropy of the relativistic cosmic ray electrons.

We further note that the instability occurs only when relativistic particles are present and arises because a fast particle's mass differs from its rest mass.

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Appendix

Let

$$S = \sum_{n=-\infty}^{\infty} \frac{J_n^2(a)}{(n^2-b^2)} \tag{A1}$$

It can be shown (Watson, 1952) that for all μ and ν such that $Re(\mu+\nu) > -1$ we can write

$$J_\mu(a) J_\nu(a) = 2\pi^{-1} \int_0^{\pi/2} J_{\mu+\nu}(2a\cos\theta) \cos(\mu-\nu)\theta \, . d\theta \quad . \tag{A2}$$

Hence when $\mu = n = -\nu$ and n is integer we see that

$$J_n^2(a) = (-1)^n 2\pi^{-1} \int_0^{\pi/2} J_0(2a\cos\theta) \cos 2n\theta \; d\theta \quad . \tag{A3}$$

Substituting for (A3) in (A1) gives

$$S = 2\pi^{-1} \int_0^{\pi/2} J_0(2a\cos\theta) \sum_{n=-\infty}^{\infty} \frac{(-1)^n \cos 2n\theta}{(n^2-b^2)} \; d\theta \quad . \tag{A4}$$

It can easily be shown that

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n \cos 2n\theta}{(n^2-b^2)} = -\pi b^{-1} \operatorname{cosec}(\pi b) \cos(2b\theta) \quad . \tag{A5}$$

Making use of (A5) in (A4) we see that

$$S = -2b^{-1}\operatorname{cosec}(\pi b)\int_0^{\pi/2} J_0(2a\cos\theta)\cos 2b\theta \, d\theta \quad .$$

Upon using (A2) in the above equation we see that

$$S = -\pi b^{-1}\operatorname{cosec}(\pi b)\, J_b(a)\, J_{-b}(a) \quad ,$$

which is the required result.

References

Gardner, F. F. and Davies, R. D., 1965, Aust. J. Phys. (in press).

Greison, K. 1956, "Progress in Cosmic Ray Physics," Vol. 3, Chapter 1, North Holland Pub. Co., Amsterdam.

Jackson, J. D., 1958, Space Technology Laboratories Report, GM-TR-0165-00535.

Penrose, O., 1960, Phys. Fluids 3, 196.

Watson, G. N., 1952, "The Theory of Bessel Functions," Cambridge University Press.

Westerhout, G., 1957, Bull. Astron. Inst. Netherlands, 13, 201 (No. 475).

Wilson, R. W., 1963, Astrophys. J. 137, 1038.

Figure Caption

Penrose criterion for instability for varying degrees of anisotropy.

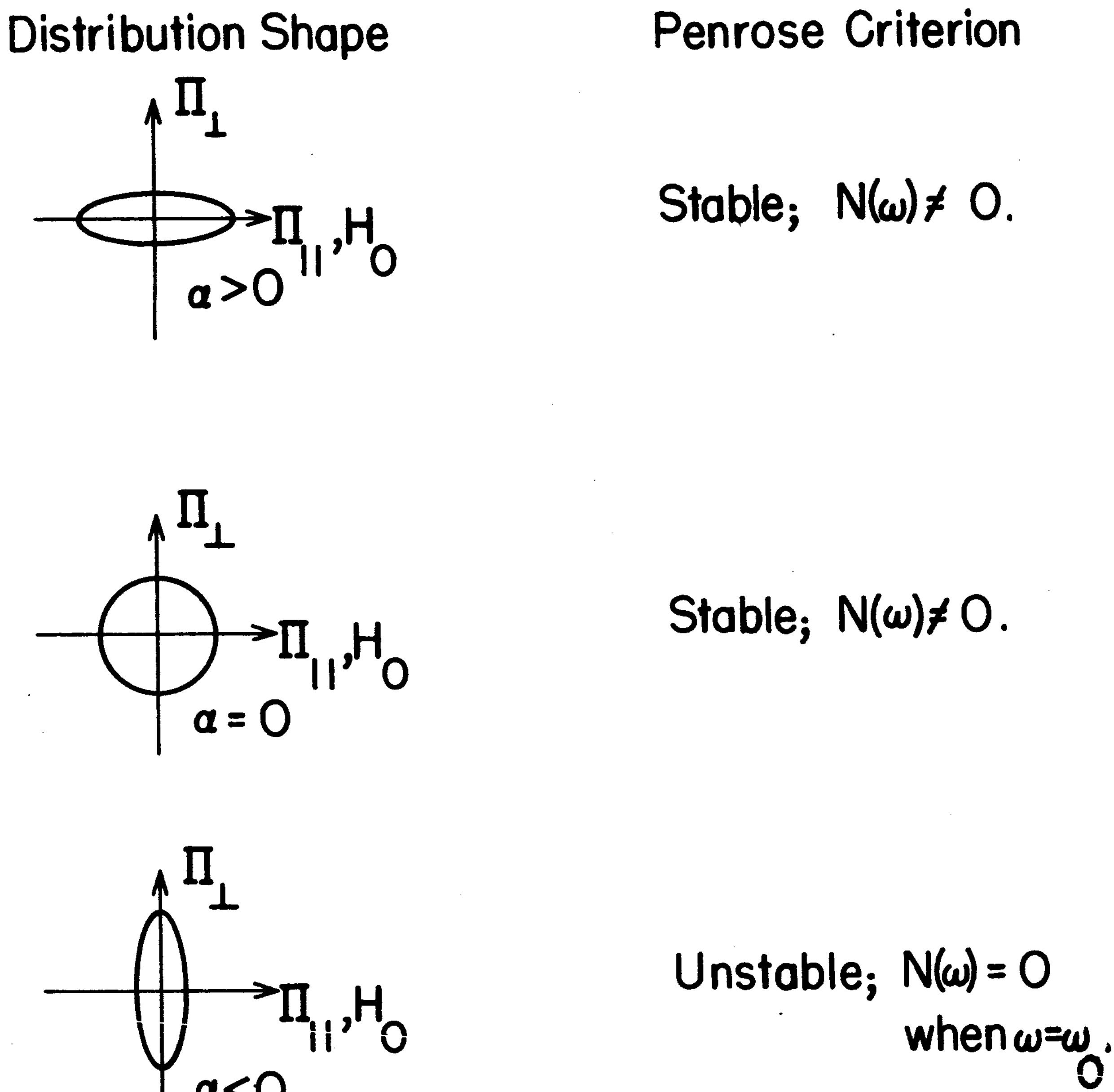
Distribution Shape
Penrose Criterion
$\Pi_\perp$
$\Pi_{\parallel}, H_0$
$\alpha > 0$
Stable; $N(\omega) \neq 0$.
$\Pi_\perp$
$\Pi_{\parallel}, H_0$
$\alpha = 0$
Stable; $N(\omega) \neq 0$.
$\Pi_\perp$
$\Pi_{\parallel}, H_0$
$\alpha < 0$
Unstable; $N(\omega) = 0$
when $\omega = \omega_0$.

Fig. 1